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Evidence of irreducibility of two-particle state space in elastic e-e-scattering ‡

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Abstract.

The S-matrix of e-e-scattering contains a projection operator that projects incoming separable product states onto entangled two-electron states. In this projection operator the empirical value of the fine-structure constant alpha acts as a weight factor. When the structure of the two-particle state space is known, a theoretical value of the weight factor can be calculated. It is shown that for an irreducible two-particle representation of the Poincaré group, the theoretical weight factor matches the empirical value of alpha. The empirical value of alpha, therefore, provides experimental evidence that the state space of two interacting electrons has the simple structure of an irreducible two-particle representation of the Poincaré group.

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> "It has been a mystery ever since it was discovered more than fifty years ago, and all good theoretical physicists put this number up on their wall and worry about it." Richard P. Feynman

1. Introduction

The standard model considers the electromagnetic interaction as the result of a fundamental gauge symmetry of nature, which requires a gauge field that couples e.g. to the electron field. Despite of its outstanding success, the principle of gauge symmetry has not been able to explain the strength of the coupling to the gauge field. In other words, it permits any value of the coupling constant, which is in clear contradiction to empirical facts. There is no question that it would be preferable to have a more *restrictive* principle of gauge invariance, which permits only certain values of the coupling constants, which then could be checked against empirical values.

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As long as gauge invariance is considered as a first principle, which cannot be deduced from more basic principles, we cannot expect to find such restrictions from within the framework of the standard model. However, there is an obvious alternative to gauge invariance as a first principle. Since gauge invariance determines the structure of the field equations of quantum electrodynamics (QED), also the reverse is true: The field equations of QED *define* a symmetry that can be expressed by invariance with respect to gauge transformations. Following this idea, we may ask the following question: Is there some as yet unexplored (mathematical or physical) element within our present description of the electromagnetic interaction (and possibly of other interactions), that *causes* gauge invariance, but, at the same time, *restricts* the value of the coupling constant to the gauge field?

The following will pursue this question and show that there is, in fact, a well-known principle that can be made responsible for the gauge symmetric structure of the equations of QED and, at the same time, restricts the value of the electromagnetic coupling constant to its empirical value.

2. Fine-structure constant and two-particle state space

The interaction term of QED is given by

$$e : \bar{\psi}(x)\, \gamma^\mu\, \psi(x) : A_\mu(x)\ . \tag{1}$$

The perturbation approach to quantum electrodynamics, following Epstein and Glaser (cf. e.g. [2]), defines the two-point distribution, built from this perturbation term,

$$D(x_1, x_2) = \ e^2 : \bar{\psi}(x_1)\, \gamma^\mu\, \psi(x_1) : A_\mu(x_1)\ : \bar{\psi}(x_2)\, \gamma^\nu\, \psi(x_2) : A_\nu(x_2)\ . \tag{2}$$

With the field operators of the electron/positron field

$$\psi(x) = (2\pi)^{-3/2} \int d\mathbf{p}\ \left(b_s(\mathbf{p}) u_s(\mathbf{p})\, e^{-ipx} + d_s(\mathbf{p})^\dagger v_s(\mathbf{p})\, e^{ipx}\right) \tag{3}$$

and of the electromagnetic field (ignoring the fact that $A^0(x)$ is usually defined in a slightly different way to ensure Lorentz covariance)

$$A_\mu(x) = (2\pi)^{-3/2} \int \frac{d\mathbf{k}}{\sqrt{2k_0}} \left(a_\mu(\mathbf{k}) e^{-ikx} + a_\mu(\mathbf{k})^\dagger e^{ikx}\right) \tag{4}$$

the corresponding two-point S-matrix

$$S_{12} = \int dx_1 dx_2\, D(x_1, x_2) \tag{5}$$

evaluates to several terms of the structure (all c-number factors are replaced by "...")

$$\int d\mathbf{p_1} d\mathbf{p_2} d\mathbf{k_1} d\mathbf{k_2}\ \ldots\ b^\dagger(\mathbf{p_1} + \mathbf{k_1})\, \gamma^\mu\, b(\mathbf{p_1})\ b^\dagger(\mathbf{p_2} - \mathbf{k_2})\, \gamma_\nu\, b(\mathbf{p_2})$$
$$\times\quad a_\mu(\mathbf{k_1})\, a_\nu^\dagger(\mathbf{k_2}). \tag{6}$$

Contraction of the photon operators results in a δ-function $\delta(\mathbf{k_1} - \mathbf{k_2})$. By integrating over $\mathbf{k_1}$ we obtain

$$\int d\mathbf{p_1} d\mathbf{p_2}\, d\mathbf{k}\ \ldots\ b^\dagger(\mathbf{p_1} + \mathbf{k})\, \gamma^\mu\, b(\mathbf{p_1})\ b^\dagger(\mathbf{p_2} - \mathbf{k})\, \gamma_\mu\, b(\mathbf{p_2})\ . \tag{7}$$

Although this term contains only electron operators, its familiar interpretation is this: a gauge particle (the photon) with momentum $\mathbf{k}$ is emitted from particle 2 and absorbed by particle 1, causing transitions from $\mathbf{p_2}$ to $\mathbf{p_2} - \mathbf{k}$ and from $\mathbf{p_1}$ to $\mathbf{p_1} + \mathbf{k}$.

Mathematically this term has a more prosaic interpretation: The S-matrix element describes the transition from an "incoming" two-particle product state to an *entangled* two-particle state and then back to an "outgoing" product state.

The product state space of incoming and outgoing particles is well-known. Very little seems to be known about the state space formed by the entangled states, especially, when we consider higher orders of the perturbation series.

The entangled states are superpositions of pure (separable) product states. Therefore, they belong to a subspace of the two-particle product state space. The transition to entangled states involves a projection onto this subspace. Therefore, the operator (7) acts as a projection operator.

A projection operator onto a subspace requires integration over a complete set of states of the subspace, rather than over the full product state space. In other words, to give (7) the mathematically correct form of a projection operator, we have to restrict the integration over $\mathbf{p_1}$ and $\mathbf{p_2}$ to a sub-domain Ω of the full product momentum space $\mathbb{R}^3 \times \mathbb{R}^3$ by replacing

$$\int d\mathbf{p_1} d\mathbf{p_2} \ldots \tag{8}$$

by

$$\int_\Omega d\omega_{\mathbf{p_1},\mathbf{p_2}} \ldots . \tag{9}$$

In a second step we can try to replace the volume element $d\omega_{\mathbf{p_1},\mathbf{p_2}}$ by the element $\omega^2\, d\mathbf{p_1} d\mathbf{p_2}$ and express the boundaries of the integral by a function $\delta(\Omega)$ that is zero outside of Ω

$$\int \omega^2\, \delta(\Omega)\; d\mathbf{p_1} d\mathbf{p_2} \ldots , \tag{10}$$

where

$$\omega^2 = \frac{d\omega_{\mathbf{p_1},\mathbf{p_2}}}{d\mathbf{p_1} d\mathbf{p_2}} \tag{11}$$

sets the (infinite) domain Ω in relation to the (infinite) $\mathbb{R}^6$.

Because of the relativistic covariance of the S-matrix, ω^2 must be a constant within a given two-particle state space. ω^2 can be understood as a weight factor, associated with a certain subspace of the product state space, or, more intuitively, as a measure for the "number of states" contained in the infinitesimal volume element $d\mathbf{p_1} d\mathbf{p_2}$. In cases, where the structure of the two-particle state space is known, ω^2 should be calculable.

When inserting the weight factor ω^2 into (7), we notice that in the same position the square of the empirical electromagnetic coupling constant e, i.e. the fine-structure constant α, is inserted "by hand" to reproduce the experimental data. Hence, after we have inserted the empirical value of α, we cannot, in addition, insert the calculated

weight factor without affecting the calculated transition amplitudes. This conflict is solved, if α and the weight factor ω^2, associated with the two-electron state space, are one and the same.

Under this premise, we should be able to identify the two-particle state space of e-e-scattering, by starting from some promising state spaces, calculating the numerical value of ω^2 and comparing it to the experimentally determined value of α. When we find a coincidence

$$\omega^2 = \alpha \ \ (= e^2, \text{ with } \hbar = c = 1)\,, \tag{12}$$

we can be pretty sure that we have found the correct two-particle state space of e-e-scattering.

Now let us see, how this idea can be put into practice.

A first conclusion is that the physical two-electron state space cannot be identical to the two-electron product space, because the latter has an associated weight factor of 1, which is different from α by two orders of magnitude.

A subspace of the product space with a non-trivial weight factor, and in fact the smallest relativistic invariant one, is the space of an irreducible two-particle representation of the Poincaré group. It represents the quantum mechanically correct description of an isolated two-particle system.

In this subspace the condition (two-particle mass hyperboloid)

$$(p_1^0 + p_2^0)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2 = m_{eff}^2 \tag{13}$$

together with

$$(p_1^0)^2 - (\mathbf{p}_1)^2 = m^2 \tag{14}$$

$$(p_2^0)^2 - (\mathbf{p}_2)^2 = m^2 \tag{15}$$

holds, where m_{eff} is the effective mass that characterizes the irreducible two-particle representation.

The numerical value of the associated weight factor ω^2 will be determined in the following section.

3. Calculation of the weight factor

The calculation of this weight factor can be based on geometric properties of the momentum parameter space of an irreducible two-particle representation, as defined by (13). As indicated by (11), the weight factor relates an *infinite* domain Ω to the *infinite* $\mathbb{R}^6$. This makes a straightforward approach difficult. Therefore, we will utilize some results from the mathematical theory of homogeneous domains [3] and try to reformulate the task of determining ω^2 within a *finite* environment.

Homogeneous domains were first studied by E. Cartan in the 1930s. They have of late found their way into theoretical physics in the context of string theories. For the purpose of the following, we can understand a homogeneous domain as a parameter space, on which a Lie group acts as a symmetry group in a transitive way. Transitivity

means: all points of the homogeneous domain are obtained from a given point by application of the elements of the symmetry group. Accordingly, a quantum mechanical state space, that has been parameterized by points of a homogeneous domain, can be generated from a single state by application of its symmetry group to this state. This makes the study of symmetries of such parameter spaces a suitable approach in analysing corresponding state spaces.

Now consider the mass shell relations (14) and (15) of two (free) electrons. When these electrons are quantum mechanically described by states of a product state space, these states are determined by six independent parameters, namely the two by three components of their 3-momenta. Therefore, the state space is generated by the group SO(3,1)×SO(3,1), acting on the parameter space. Within an *irreducible* representation, condition (13) reduces the number of independent parameters from six to five. When we use (13) to determine e.g. the third component p_2^3 of the momentum of the second particle, then the parameter space of the irreducible two-particle state space can be generated by the group SO(3,1)×SO(2,1). This group can be embedded as a subgroup into SO(5,2). Since the structure of SO(5,2) is simpler than that of SO(3,1)×SO(2,1), we shall start our analysis from this group, and then apply a correction with regard to SO(3,1)×SO(2,1).

Associated with SO(5,2) is the quotient group SO(5,2)/(SO(5)×SO(2)). This quotient group is a homogeneous domain with a transitive action of SO(5,2). Therefore, this domain can serve as a natural parameterization of a state space that is invariant under the group SO(5,2).

A well-known bounded "realization" of this domain is the complex Lie ball (cf. [5])

$$D^5 = \{z \in \mathbb{C}^5; 1 + |zz'|^2 - 2\bar{z}z' > 0, |zz'| < 1\} \; . \tag{16}$$

The boundary of D^5 is given by

$$Q^5 = \{\xi = x\, e^{i\theta}; x \in \mathbb{R}^5, xx' = 1\}, \text{ with } 0 < \theta < \pi \; . \tag{17}$$

(The vector z' is the transposed of z, $\bar{z}$ is the complex conjugate of z.) SO(5)×SO(2) acts transitively on Q^5.

The Lie ball is included in the complex unit ball

$$C^5 = \{z \in \mathbb{C}^5; |zz'| < 1\} \tag{18}$$

and contains the real unit ball

$$B^5 = \{x \in \mathbb{R}^5; xx' < 1\} \; . \tag{19}$$

The complex unit ball is isomorphic to the upper half-space of $\mathbb{C}^5$, whereas the Lie ball is isomorphic to the forward light cone in $5+2$ dimensions (with imaginary "time"-coordinates). A conformal Möbius transformation can be used to map the (bounded!) complex unit ball onto the (unbounded!) upper half-space and vice versa.

On the internet a very instructive animation of the Möbius transformation can be found §. Readers not familiar with Möbius transformations may want to load the (medium resolution) video of this animation before continuing.

§ http://www.ima.umn.edu/videos/mobius.php

The complex unit ball C^5 or the Lie ball D^5 offer themselves as parameter space for a SO(5,2) symmetric system: Firstly, calculations on this finite domain are simpler that on the infinite $\mathbb{C}^5$. Secondly, the Lie ball is optimally adjusted to the geometry of the parameter space, as defined by (13), because it immediately reflects its spherical symmetry properties.

For the following let us assume that we have restricted the range of integration in (7) to the subspace of an irreducible two-particle representation, as indicated in (9), and have parameterized the corresponding parameter subspace by coordinates on D^5.

Then we can separate the integral into a spherical integral over the surface Q^5 of D^5 and a second integral over the radial direction of D^5. The spherical part is given by

$$\int_{Q^5} \frac{d^4\xi}{V(Q^5)} \ \cdots \ . \tag{20}$$

The integral is normalized by the volume of Q^5, since a projection operator adds up states, not volume elements of the parameter space.

The integration over the phase θ (cf. (17)) can immediately be done and delivers a factor of π, because the integrand does not explicitly depend on θ. In combination with the contribution from the Lie ball, corresponding to the lower half-space of $\mathbb{C}^5$, we obtain a factor of 2π. (Phase θ corresponds to the orientation of the two-particle energy-momentum vector in the p_1^0-p_2^0-plane, but p_1^0 and p_2^0 can be eliminated by using the mass shell relations (14) and (15). Therefore, the integrand does not depend on θ.) After this integration, the remaining parameters form a real sphere, which is a subspace of Q^5, and $d^4\xi$ is replaced by d^4x with real parameters x.

Next we have to add the integration in radial direction of D^5. More precisely, we want to extend the four-dimensional infinitesimal volume element d^4x to a *Cartesian* infinitesimal volume element in *five* dimensions.

Mapping a spherical volume to a rectangular one includes a step that is known as "quadrature of the circle". (As an example: the volume of the unit ball equals the volume of a cube with edge length $\sqrt[3]{4\pi/3}$.)

Furthermore, we have to keep in mind that the volume element of the original integral (7) is isotropic with respect to all six directions of $\mathbb{R}^3 \times \mathbb{R}^3$. Since we want to obtain the weight factor as a factor to this isotropic volume element, we have to make sure that also the new volume element is isotropic.

Consider the formula that relates the volume of a Lie ball D_R^5 with radius R to the volume of the unit Lie ball D^5

$$V(D_R^5) = R^5\, V(D^5)\ . \tag{21}$$

When we project the volume of D^5 onto the real ball B^5 with surface S^4, then $V(D_R^5)$ can be expressed by the integral

$$V(D_R^5) = 5\int_0^R dr \int_{S^4} r^4\, v^4\, d\omega_x\,, \ \text{with } v = V(D^5)^{\frac{1}{4}}\,, \ \text{if} \int_{S^4} d\omega_x = 1\,. \tag{22}$$

A rectangular volume with the same numerical value is given by

$$5 \int_0^R dr \int_0^r v\, dx_1 \int_0^r v\, dx_2 \int_0^r v\, dx_3 \int_0^r v\, dx_4 \,. \tag{23}$$

This integral is an analogue to the "quadrature of the circle". Unfortunately, it maps the volume of the Lie ball not to a cube, but to the cuboid

$$R^5 \times (1 \times v \times v \times v \times v) \,. \tag{24}$$

An infinitesimal volume element of (22) at $(x, r) = (0, 1)$ is accordingly mapped to

$$(dr, v\, dx_1, v\, dx_2, v\, dx_3, v\, dx_4) \,. \tag{25}$$

Consequently, to obtain an isotropic volume element, the coordinate in radial direction must be scaled according to

$$dr = v\, dx_5 = V(D^5)^{\frac{1}{4}}\, dx_5 \,. \tag{26}$$

Therefore, to extend the 4-dimensional volume element d^4x to a 5-dimensional Cartesian isotropic volume element, we have to multiply d^4x by the right hand side of (26).

The fifth dimension also adds a factor to the normalization of the projection operator, but for the Lie ball of radius 1 this factor is equal to 1, as can be seen by inspection of (22) or (23).

Up to now we have integrated over the full SO(5,2)-symmetric D^5; but remember that the generating group of the two-particle state space is not the group SO(5,2) but SO(3,1)×SO(2,1). This means, a "rotation" from the momentum parameters of the first particle to the second is not a valid symmetry operation and, therefore, parameter sets, generated in this way, should not contribute to the integral. On the other hand, one can easily see by inspection of (13), that such operations generate valid parameter combinations that identify states of the same irreducible representation. Consequently, the integration over the *full* D^5 will add *multiple copies* to each state. This is not a problem, since in a projection operator we integrate over all states of the representation. We only have to restore the correct weight of the states by dividing by the multiplicity factor.

Let us, for a moment, replace the continuous group operations by discrete ones. Then count all points of the parameter space that are generated by five orthogonal rotations of SO(5). Divide by the number of points obtained from the four independent rotations contained in the generating group SO(3,1)×SO(2,1). This delivers the multiplicity factor. Group theory tells us that this number is identical to the number of elements of the quotient group SO(5)/SO(4). This quotient group SO(5)/SO(4) is known to be isomorphic to the (real) unit sphere S^4 in five dimensions (see e.g. [4]). In going back to continuous group operations, sums over discrete points are replaced by volumes. Therefore, the integral has to be corrected by a factor of $1/V(S^4)$. This adds the factor $1/V(S^4)$ to the volume element d^5x.

So far we have ignored spin degrees of freedom. When we include spin, then the set of orthonormal two-particle states in the projection operator is extended by factor

4, resulting from 2×2 spin states of the electron states. This adds a factor of 4 to the "number of states" within the infinitesimal volume element d^5x.

Collecting all factors results in a total weight factor ω^2 of

$$8\pi\, V(D^5)^{\frac{1}{4}} \,/\, (V(S^4)\, V(Q^5)) \,. \tag{27}$$

In collecting its contributions, we have not found any dependency on the momentum parameters. This confirms that the weight factor is a constant.

Now we can apply a multidimensional Möbius transformation $M : x \to p$

$$p = \left(\frac{x_1}{1-x_5}, \frac{x_2}{1-x_5}, \frac{x_3}{1-x_5}, \frac{x_4}{1-x_5}, \frac{x_5}{1-x_5} \right) \tag{28}$$

and map the real unit ball B^5, subspace of D^5, into the upper half-space of $\mathbb{R}^5$. The Möbius transformation (28) is a conformal mapping. The proof is by writing down (28) for an infinitesimal cube. Therefore, the isotropic infinitesimal volume element, which has been constructed on B^5, is mapped onto an isotropic Cartesian volume element $\omega^2 d^5p$ in $\mathbb{R}^5$.

A more intuitive, though less elegant way would be to replace the unit Lie ball by a Lie ball with radius R and then let $R \to \infty$.

Finally, the five-dimensional volume element $\omega^2 d^5p$ has to be embedded into the six-dimensional Euclidean momentum space $\mathbb{R}^3 \times \mathbb{R}^3$ by identifying the five dimensions of the volume element with five directions of the momentum space. The sixth component p_n of the momentum then corresponds to a direction perpendicular to the surface defined by (13). Therefore, the function $\delta(\Omega)$ in (9) is in fact an ordinary δ-function, and an integration over p_n delivers a factor of 1.

The volumes $V(D^5)$ and $V(Q^5)$ have been calculated by L. K. Hua [5]. With

$$V(Q^5) = \frac{8\pi^3}{3} \tag{29}$$

$$V(D^5) = \frac{\pi^5}{2^4\, 5!} \tag{30}$$

$$V(S^4) \,= \frac{8\pi^2}{3} \tag{31}$$

we obtain

$$\frac{9}{8\pi^4} \left(\frac{\pi^5}{2^4\, 5!} \right)^{1/4} \;=\; \frac{9}{16\pi^3} \left(\frac{\pi}{120} \right)^{1/4} \;=\; 1/137.03608245 \,. \tag{32}$$

This value agrees up to a factor of 0.9999995 with the experimental (low energy) value of α, which is 1/137.035 999 084(51) [6].

Formula (27), which approximates the fine-structure constant by volumes of certain bounded homogeneous domains, was discovered forty years ago by the Swiss mathematician Armand Wyler [7]. Unfortunately, Wyler was not able to put his observation into a convincing physical context. In particular, Wyler was not aware of the relation to irreducible two-particle representations. Therefore, his work was criticized (Robertson [8]) and, in the following decades, it was considered as fruitless numerology.

The extremely close agreement of ω^2 with the (low energy) empirical value of α delivers a strong experimental indication that the (low-energy) physical two-particle state space of elastic e-e-scattering matches an irreducible two-particle representation (of identical, massive, half spin particles) of the Poincaré group. Since H. Joos's paper on representations of the Lorentz group [9] these representations are generally known.

Wyler's formula defines a geometrical factor that relates an irreducible two-particle representation of the Poincaré group to a corresponding two-particle product representation, just as π relates the circumference of a circle to its diameter. To relate this geometrical factor to the empirical fine-structure constant, we have to keep in mind that the latter is determined experimentally. Therefore, not only the contribution from *elastic* e-e-scattering, which is the lowest order of the perturbation series, but also all higher orders, contributing to e-e-scattering, determine its value. The accumulation of these contributions is described by the renormalization group. It leads to an energy dependent "effective" coupling constant (running coupling constant). At low energies and depending on the experimental setup, non-elastic contributions of "infrared photons" can be kept well under control. Therefore, the fine structure constant, obtained from low-energy e-e-scattering comes close to the value obtained from elastic scattering. This explains the success of Wyler's formula in reproducing the empirical value of alpha.

4. Conclusions

The agreement of the empirical value of alpha with the calculated coupling constant in elastic e-e-scattering, provides experimental evidence that the S-matrix contains a projection onto an *irreducible two-particle representation* of the Poincaré group. With the understanding of the S-matrix as a projection operator, we can say: The S-matrix describes a transition from a separable product state of two incoming electrons (preparation) to an intermediate irreducible two-particle state (propagation) and then back to a separable product state of two outgoing electrons (analysis).

The formation of irreducible two-particle states can be understood as the manifestation of a basic quantum mechanical rule: *An isolated quantum mechanical system is described by an irreducible representation of the Poincaré group.* Therefore, the form of the S-matrix can fully be explained by elementary quantum mechanics and Poincaré invariance. On the other hand, we have not touched the gauge invariant structure of the perturbation algorithm. This means, we have traced back gauge invariance to a basic rule of quantum mechanics. However, now *gauge invariance goes together with a certain value of the coupling constant*, and we are lucky enough that this value matches the low energy value of the empirical fine-structure constant.

This new access to QED has been found by simply inverting the logical chain of reasoning: Instead of explaining QED by gauge invariance, we explain (a restrictive form of) gauge invariance by the mathematical structure of the S-matrix, which in turn is understood as the result of a basic quantum mechanical rule. Both interpretations understand the photon field as an auxiliary means, either to ensure gauge invariance

or to generate irreducible two-particle states. In the traditional interpretation of QED the entanglement of the two-particle state is *caused* by the exchange of virtual gauge particles. By inverting the logical chain, we can say that the entanglement of the two-particle state *permits* a description in terms of virtual gauge particles. Entanglement, in turn, is a natural property of an irreducible two-particle state space, because, if some states were not entangled, they could be used to generate a (reducible) product state space, in contradiction to irreducibility.

Our analysis of QED has been based on the lowest order of the perturbation series. Higher orders are obtained by iterating the lowest order. Because of this, higher orders are mathematically completely determined by the lowest order. Therefore, all what has been said about the lowest order applies similarly to higher orders.

We may replace the traditional illustration of the perturbation algorithm (in terms of continuous creation and annihilation of virtual particles from the vacuum), by a new one: *Electromagnetic interaction results from continuous virtual transitions between separable product states and irreducible two-particle states.* However, we should beware of taking this description, as well as the traditional one, too literally, because both reflect rather (an interpretation of) the mathematical structure of the algorithm than real physical processes.

I should add some remarks concerning the history of Wyler's formula. At that time, when Wyler found this expression, his favourite subject was: "the various components of the boundaries of complex domains associated with Lie groups" [10]. He observed that an expression, derived from a domain that is related to Maxwell's equations, delivered the numerical value of the fine-structure constant. He published his finding in the hope, that "if he piqued the interest of the physics community, there might be more study of his favorite subject" [10]. Unfortunately, the physics community neither understood his intention nor his mathematics.

One of the main objections raised was, that Wyler used certain bounded spaces with a radius equal to 1. It was argued (Robertson [8]), that "there is no known reason for setting r = 1", and it was suspected that a different radius would yield a different value for α. Another point of critics was that Wyler could not clearly specify how the forth-root factor entered the calculation.

From the derivation above, it should be clear that Wyler's formula does not depend on the radius of the Lie sphere. The reason is that ω^2 is defined as the *quotient* of two volume elements (cf. (11)) on the surface of the two-particle mass hyperboloid. Whether we map these volume elements to a Lie sphere with radius 1 or any other radius or do not map it at all, does not have any influence on this quotient. Speaking generally, the volumes in Wyler's formula are not the outcome of the mapping onto the Lie sphere, but rather reflect the internal geometrical structure of the homogeneous domain SO(5,2)/(SO(5)×SO(2)), which is independent of any mapping.

What really counts, is the conformity of the mapping, because we compare different direction within a volume element. This concerns the step leading to the SO(5)/SO(4) factor and especially to the forth-root factor in (26), which is obtained from inspection

of (24) and (25). This expression clearly shows the importance of conformity and, at the same time, the independency of this factor of the radius R.

In an answer to Robertson's objections Robert Gilmore [11] wrote:
"Wyler's work has pointed out that it is possible to map an unbounded physical domain - the interior of the forward light cone - onto the interior of a bounded domain on which there also exists a complex structure. This mapping should prove of immense calculational value in the future."

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